§2.1 Matrix Operations

Notation

For A an $m \times n$ matrix we might write

$$A = [a_1 | \cdots | a_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$$

column notation   entry notation

Here a_{ij} is the entry in the i^{th} row and j^{th} column.

- An $n \times n$ matrix is sometimes called a <u>square matrix</u>.

- A <u>diagonal matrix</u> is an $n \times n$ matrix whose non-diagonal entries are zero. In other words if $A = (a_{ij})$

$$a_{ij} = 0 \text{ if } i \neq j$$

<u>example</u>

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

- A <u>zero matrix</u> is a matrix all of whose entries are zero.

- We say two matrices are <u>equal</u> if they have the same size and their entries are the same.

Matrix Addition/Subtraction

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of the same size, addition and subtraction are defined component-wise

$$A + B = (a_{ij} + b_{ij}) \qquad A - B = (a_{ij} - b_{ij})$$

For example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Scalar Multiplication

If A is any sized matrix and c a real number scalar multiplication is also defined componentwise. If $A = (a_{ij})$, $c \cdot A = (c \cdot a_{ij})$ (multiply each entry)

For example

$$3 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Properties of Addition and scalar multiplication

Let A, B, C be $m \times n$ matrices (all same size) and

r, s real numbers

1) $A + B = B + A$

2) $(A+B) + C = A + (B+C)$

3) $A + 0 = A$ (Here 0 denotes the $m \times n$ zero matrix)

4) $r(A+B) = rA + rB$

5) $(r+s)A = rA + sA$

6) $r(sA) = (rs)A$

Matrix Multiplication

We have seen column vector multiplication in §1.4

If $A = [a_1 | \cdots | a_n]$ is an $m \times n$ matrix and

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is an $n \times 1$ column vector,

$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

which is an $m \times 1$ column vector

Defn

If A is an $m\times n$ matrix and B is an $n\times p$ matrix, we define matrix multiplication as follows.

Write $b_1, \dots, b_p$ for the columns of B

$$AB = A\left[b_1 | \cdots | b_p\right] = \left[Ab_1 \,|\, Ab_2 \,|\, \cdots \,|\, Ab_p\right]$$

where Ab_i $(1 \le i \le p)$ is the column vector multiplication previously mentioned.

- Notice each Ab_i $(1 \le i \le p)$ is an $m\times 1$ column vector.
- Thus AB is an $m\times p$ matrix

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} = \underset{m\times p}{AB}$$

have to be equal

Dimensions of new matrix

- In general, for this multiplication to make sense for $A\cdot B$, we must have

$$\#\text{ columns of } A = \#\text{ rows of } B$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b_1 \quad b_2$$

what is AB?

$$AB = [\, Ab_1 \mid Ab_2 \,]$$

$$= \left[A\begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\middle|\; A\begin{bmatrix} -1 \\ 3 \end{bmatrix} \right]$$

$$= \left[\begin{array}{c|c} 1+4 & -1+6 \\ 4+10 & -4+15 \\ 3+12 & -3+18 \end{array}\right]$$

$$= \begin{bmatrix} 5 & 5 \\ 14 & 11 \\ 15 & 15 \end{bmatrix}$$

which is 3×2

since A is 3×2
B is 2×2

Warning!

Matrix Multiplication is not commutative! $AB \neq BA$

Sometimes BA isn't even defined even if AB is!

For instance in the example above $\underset{2\times 2}{B} \cdot \underset{3\times 2}{A}$ does not make sense!

$2 \neq 3$

Remark

Not only do the sizes of A and B dictates the size of AB, they also dictate the entries!

Example

$$A = \begin{bmatrix} 1 & -2 & 8 & 9 \\ 0 & 5 & 4 & 2 \\ -1 & 7 & 6 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ -6 & 2 \\ 7 & 5 \\ 0 & -9 \end{bmatrix}$$

What is the (3,2) entry of AB?

Solution: We need only look at the 3rd row of A and the 2nd row of B

$$\begin{bmatrix} -1 & 7 & 6 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \\ -9 \end{bmatrix} = 1 + 14 + 30 - 27$$

$$= \boxed{18}$$

But why is this?

$$AB = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} = \begin{bmatrix} Ab_1 & \begin{matrix} R_1 b_2 \\ R_2 b_2 \\ R_3 b_2 \end{matrix} \end{bmatrix}$$

(3,2) entry here

2nd column

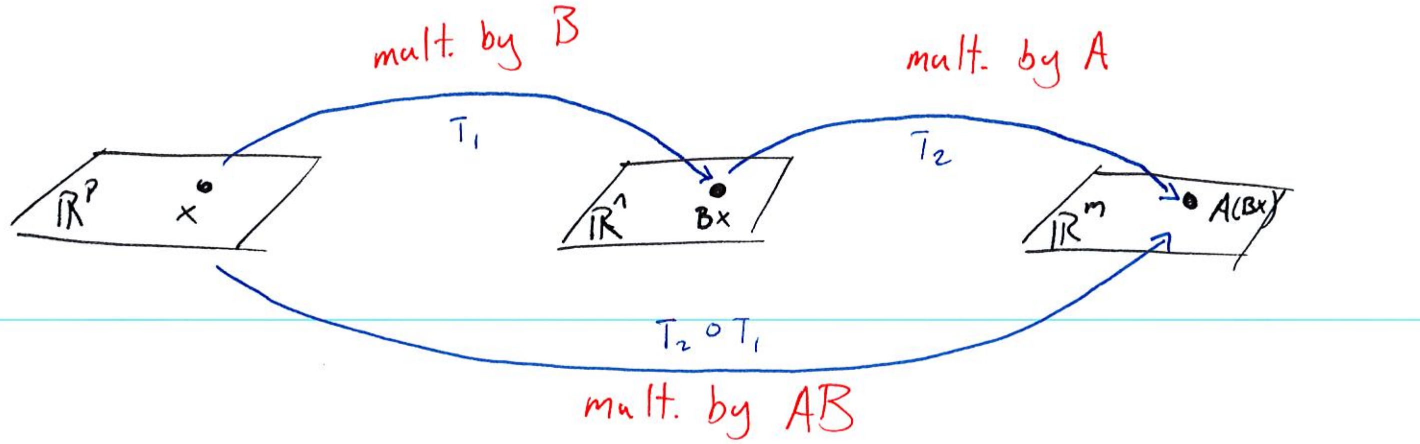
where R_1, R_2, R_3 are the columns of A

Why do we care?

Suppose A is $m \times n$ and B is $n \times p$

If $T_1 : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ and $T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

are linear transformations given by $T_1(x) = Bx$ and $T_2(y) = Ay$, then AB represents the composition of T_1 and T_2, $T_2 \circ T_1$



Exercise

1) Show that the matrix associated to the transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points counterclockwise by $2\pi/3$ radians is $\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$.

2) Show that the matrix associated to the transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points across the horizontal x_1-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

3) Compute $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$. How does this compare to the result of the example from 2/8? Why is this?